ISI – Bangalore Center – B Math - Physics III – End Term Exam
Date: 15 November 2019. Duration of Exam: 3 hours
Total marks: 80
Answer ALL Questions

Q1. [DO ANY THREE PARTS of Q1. Total Marks: 3x5=15]

1a.) Suppose we have an electrostatic charge configuration of finite spatial extent. Choose a coordinate system so that the charges are all within a sphere of finite radius centered at the origin. Suppose we express the potential *V*(*r*) far away from the charge configuration as
$V(r) = \frac{A}{r^{1/2}} + \frac{B}{r} + \frac{C}{r^2} + ...$ where A, B, C are constants.
Between *A* and *B*, which one is always zero and why? When are both A and B zero?

1b.) Which of the following HAVE TO BE always true, as per Maxwell's equations,
$\oint_C \vec{A}.dl =$ Magnetic flux through a surface of which *C* is the boundary,
Or,
$\nabla^2 \vec{A} = -\mu_0 \vec{j}$?
Here $\vec{A}$ is the magnetic vector potential.

Derive from Maxwell's equations, the result that you think is always true and explain briefly in two or three sentences why the other one is not always true.

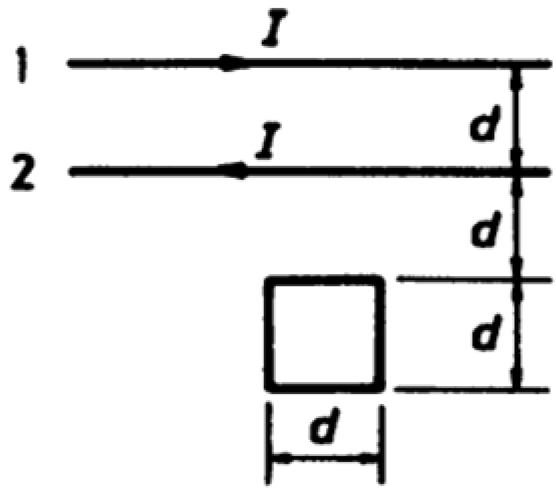
1.c) The electromagnetic energy density is given by $u_{em} = \frac{1}{2}(\varepsilon_0 |\vec{E}|^2 + \frac{1}{\mu_0} |\vec{B}|^2)$.
Suppose we compute $\iiint_{charge\_free\_region} u_{em} d^3x$, the total electromagnetic energy for a charge free region. Explain why $\frac{d}{dt}\left(\iiint_{charge\_free\_region} u_{em} d^3x \right)$ may not be zero and state without derivation how this is related to the Poynting vector $\vec{S}$.

1.d) Suppose we have charges distributed on a surface, described by a given surface charge density. We know that if the charges are kept static, then $\vec{\nabla} \times \vec{E} = 0$, which leads to the boundary condition $\vec{E}^{above\_surface}_{parallel} = \vec{E}^{below\_surface}_{parallel}$.

Does that equation remain true in non static situation? If yes, give a proof. If not, give a counterexample.

Q2. [Total Marks: 2+8=10]

Two infinite parallel wires separated by a distance d carry equal currents I in opposite directions, with I increasing at the rate of $\frac{dI}{dt}$. A square loop of wire of length d on each side, lies in the plane of the wires at a distance d from one of the parallel wires, as shown in the figure below.



2a.) Is the induced current clockwise or anticlockwise? Justify your answer.

2b.) Show that the emf induced in the square loop is given by $-\frac{\mu_0 d\alpha}{2\pi}\frac{dI}{dt}$ where α is a constant. Find the value of α.

Q3. [Total Marks: 3+4+8=15]

The vector magnetic potential $\vec{A}$ in a magnetostatic system satisfies the equation $\nabla^2\vec{A} = -\mu_0\vec{j}$.

3a.) Using the fact that the formal solution of vector potential $\vec{A}$ is given by $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\vec{j}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}d^3r'$ show that the magnetic vector potential for a uniform surface current $\vec{K} = K\delta(z)\hat{x}$, where K is a constant, can only have components in the x direction.

3b.) Using the result of 3a.) show that the magnetic field can only be in the y direction.

3c.) Show that the magnetic field is given by

$$\mathbf{B} = \begin{cases} +(\mu_0/2)K\,\hat{\mathbf{y}} & \text{for } z < 0, \\ -(\mu_0/2)K\,\hat{\mathbf{y}} & \text{for } z > 0. \end{cases}$$

Q4. [Total marks: 5+5+6+4=20]

The electric potential of some static charge configuration is given by the expression $V(\vec{r}) = A\frac{e^{-\lambda r}}{r}$ where A and λ are constants with appropriate dimensions.

4a.) Find the electric field everywhere.

4b.) Using Gauss's law, find the charge inside a sphere of radius *r*. Hence calculate the total charge in the system.

4c.) Show that the charge density is given by
$\rho = \epsilon_0 A(4\pi\delta^3(\mathbf{r}) - \lambda^2 e^{-\lambda r}/r)$.
[Suggestion: This part is calculation intensive. Whether you prove this or not, you can use this result in part d.)]

4d.) Integrate the charge density to find the total charge and show that this agrees with the result you got in part b.)

Q5. [Total Marks: 5x4=20]

Suppose that the electric field of an electromagnetic wave in a region of vacuum is given by

$\vec{E} = E_0(\sin(kz - wt)\vec{i} + \sin(kz + wt + \delta)\vec{j})$, where E_0 is a constant, and $c = w/k$.

5a.) Starting from the appropriate Maxwell's Equation, calculate the corresponding magnetic field. [Hint: Use linearity of Maxwell's equations to connect this problem to known results]

5b.) What is the energy per unit area per unit time (the Poynting vector $\vec{S}$) transported by this wave?

5c.) Calculate the time average of the Poynting vector over a cycle $T = \frac{2\pi}{w}$. Briefly interpret the result you get.

5d.) If instead the electric field was $\vec{E} = E_0(\cos(kz - wt)\vec{i} + \cos(kz - wt + \delta)\vec{j})$, what would be time average of the Poynting vector? Detailed calculation not needed. You can use standard results for linearly polarized light, if needed.

============The results given below may be useful=============

$\nabla(fg) = f\nabla g + g\nabla f$, $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$, where f,g are scalar functions , $\boldsymbol{A}$ is vector field

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}}$$

in spherical coordinates.

$$\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right) = 4\pi\delta^3(\mathbf{r})$$

Maxwell's Equations:

$$\begin{aligned}
&\text{(i)} && \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \\
&\text{(ii)} && \nabla \cdot \mathbf{B} = 0 \\
&\text{(iii)} && \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\
&\text{(iv)} && \nabla \times \mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}
\end{aligned}$$

Poynting Vector:

$$\mathbf{S} \equiv \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$$

For a plane monochromatic Electromagnetic wave:
$\vec{S} = c\varepsilon_0 |\vec{E}|^2$, where $\vec{E}$ is the associated electric field.